

Spinor Theory of Gravity

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The proposal of this work is to provide an answer to the following question: is it possible to treat the metric of space-time - that in General Relativity (GR) describes the gravitational interaction - as an effective geometry? In other words, to obtain the dynamics of the metric tensor $g_{\mu\nu}$ as a consequence of the dynamics of other fields. In this work we will use a slight modification of the non-linear equation of motion of a spinor field proposed some years ago by Heisenberg, although in a completely distinct context, to obtain a field theory that provides a framework equivalent to the way GR represents the gravitational interaction. In particular we exhibit a solution of the equations of motion that represents the gravitational field of a compact object and compare it with the corresponding Schwarzschild solution of General Relativity.

I. INTRODUCTION

By the Equivalence Principle [1] the gravitational interaction may be described as a modification of the geometry of space-time. In the General Relativity theory this idea was implemented by assuming that there exists a unique geometrical structure which acts on all forms of matter and energy (including the gravitational one) in the same way. Moreover, Einstein postulated an equation of motion to describe the evolution of the geometry under the very natural assumption that the gravitational field (identified with the geometrical structure) should have a dynamics of its own. These two parts of the description are related but independent. Taken together, the Equivalence Principle and Einstein's equation, constitute the basis of a successful program of a theory of gravity.

Is this the unique way to deal with the universality of gravitational processes? In this work we propose a new way to implement the Equivalence Principle in which the geometry acting on matter is not an independent field, and as such does not possess its own dynamics. Instead, it inherits one from the dynamics of two fundamental spinor fields Ψ and Υ which are responsible for the gravitational interaction and from which an effective geometry will be constructed. The nonlinear character of gravity should be present already at the most basic level of these fundamental structures. It seems natural to describe this nonlinearity in terms of the invariants constructed with the spinor fields. The simplest way to build a concrete model is to use the standard form of a contraction of the currents of these fields, e. g. $J_\mu J^\mu$ to construct the Lagrangian of the theory. This will lead us to deal with Heisenberg's equation of motion, which has precisely this form, although it originated in a completely different context and is written in terms of others invariants.

We assume that these two fields (which are half-integer representation of the Poincaré group) interact universally with all other forms of matter and energy. As a consequence, this process can be viewed as nothing but a change of the metric of the space-time. In other words we shall show that the influence of these spinor fields on matter/energy is completely equivalent to a modification

of the background geometry into an effective Riemannian geometry $g_{\mu\nu}$. In this aspect this theory agrees with the idea of General Relativity theory which states that the Equivalence Principle implies a change on the geometry of space-time as a consequence of the gravitational interaction. However, the similarities between the Spinor Theory of Gravity and General Relativity stop here.

To summarize let us stress the main steps of this new program.

- a. There exist two fundamental spinor fields – which we will name Ψ and Υ ;
- b. These fields obey the nonlinear Heisenberg equation of motion;
- c. The fields Ψ and Υ interact universally with all forms of matter and energy;
- d. As a consequence of this coupling with matter, this universal interaction produces an effective metric;
- e. The dynamics of the effective metric is already contained in the dynamics of Ψ and Υ : the metric does not have a dynamics of its own, but inherits its evolution through its relation with the fundamental spinors;
- f. We present a particular example of the effective metric in the case of a compact spherically static object, like a star. We compare it with the analogous analysis in the case of General Relativity.

Before entering the analysis of these questions let us briefly comment our motivation. As we shall see, the present proposal and the theory of General Relativity have a common underlying idea: the characterization of gravitational forces as nothing but the effect on matter and energy of a modification of the geometry of space-time. This major property of General Relativity, remains unchanged. The main difference is about the dynamics that this geometry obeys. In GR the dynamics of the gravitational field depends on the curvature invariants; in the Spinor Theory of Gravity such a specific dynamics simply does not exist: the geometry evolves in space-time according to the dynamics of the spinors Ψ and Υ . The metric is not a field of its own, it does not have an independent reality but is just a consequence of the universal coupling of matter with the fundamental spinors. The motivation of walking down only half of Einstein's path

to General Relativity is to avoid certain known problems that still plague this theory, including its difficult passage to the quantum world; the questions put into evidence by astrophysics involving many discoveries such as the acceleration of the universe, the problems requiring dark matter, dark energy. Up to now, only highly speculative ideas have appeared to solve such problems, culminating with a plethora of proposals of scalar fields with negative energies. Also, the possibility of relating gravitational phenomena to elementary fields, usually identified to matter fields, may present new possibilities of future developments on a new road towards a unified program. With this in mind, we will present a toy model to discuss such an idea.

In section II we present the mathematical background used in the paper. In particular we analyze the standard form of the internal connection needed to produce a covariant definition of the derivative of a spinor. Even in a flat manifold, like Minkowski geometry, this generalization of the derivative is needed to obtain a covariant description in a non-Euclidean coordinate system. We recall the idea that the expression of Fock-Ivanenko that displays the internal connection in terms of derivatives of the metric and of the Dirac matrices γ_μ 's is not unique. The form of Fock-Ivanenko assumes that the covariant derivative of the γ_μ 's vanishes. This possibility is allowed by the fact that the background manifold is Riemannian, but any function of the Clifford algebra could imply the same property for the metric structure. We are then led to add to the internal connection a vector U_μ . This vector is an element of the Clifford algebra and will be constructed in terms of the currents of the fundamental spinor fields. We show that this choice of the connection allows a new geometrical interpretation of the Heisenberg self-interaction of a spinor field. Then, in section V, we review the field theory formulation of General Relativity as it was described in the fifties by Gupta, Feynman [2] and others, and more recently in [3]. In this formulation the gravitational field can be described alternatively either as the metric of space-time – as in Einstein's original version – or as a field $\varphi_{\mu\nu}$ in an arbitrary unobservable background geometry, which is chosen to be Minkowski (see also [7]). We shall see that by universally coupling the spinor fields to all forms of matter and energy, a metric structure appears, in a similar way to the field theoretical description of GR. The main distinction between these two approaches concerns the status of this metric. In General Relativity it has a dynamics provided by a Lagrangian constructed in terms of the curvature invariants. In our proposal, this is not the case. The metric is an effective way to describe gravity and it appears because of the universal form of the coupling of matter/energy of any form and the fundamental spinors. In section VI we present a toy model of the dynamics of these fields. Section VII is dedicated to a particular solution of these equations in the description of the static and spherically symmetric external gravitational field generated by a massive object such as a star. We exhibit the

resulting effective metric and compare it with the similar situation in the General Relativity theory. We end with some conclusions and future perspectives.

II. DEFINITIONS AND SOME MATHEMATICAL MACHINERY

The vector and the axial currents are defined in the standard way:

$$J^\mu \equiv \bar{\Psi} \gamma^\mu \Psi$$

$$I^\mu \equiv \bar{\Psi} \gamma^\mu \gamma^5 \Psi.$$

In this paper we deal with two spinor fields Ψ and Υ . We use capital symbols to represent the currents constructed with Ψ as above and lower case to represent the corresponding terms of the spinor Υ , namely,

$$j^\mu \equiv \bar{\Upsilon} \gamma^\mu \Upsilon$$

$$i^\mu \equiv \bar{\Upsilon} \gamma^\mu \gamma^5 \Upsilon.$$

We use the standard convention and definitions (see, for instance [5]). For completeness we recall:

$$\bar{\Psi} \equiv \Psi^\dagger \gamma^0.$$

The γ^5 is hermitian and the other γ_μ obey the hermiticity relation

$$\gamma_\mu^\dagger = \gamma^0 \gamma_\mu \gamma^0.$$

The properties needed to analyse non-linear spinors are contained in the Pauli-Kofink (PK) relation. These are identities that establish a set of tensor relations concerning elements of the four-dimensional Clifford algebra. The main property states that, for any element Q of this algebra, the PK relation ensures the validity of the identity:

$$(\bar{\Psi} Q \gamma_\lambda \Psi) \gamma^\lambda \Psi = (\bar{\Psi} Q \Psi) \Psi - (\bar{\Psi} Q \gamma_5 \Psi) \gamma_5 \Psi. \quad (1)$$

for Q equal to \mathbb{I} , γ^μ , γ_5 and $\gamma^\mu \gamma_5$, where \mathbb{I} is the identity of the Clifford algebra. As a consequence of this relation we obtain two extremely important facts:

- The norm of the currents J_μ and I_μ have the same value and opposite sign.
- The vectors J_μ and I_μ are orthogonal.

Indeed, using the Pauli-Kofink relation we have, for $Q = \mathbb{I}$

$$(\bar{\Psi} \gamma_\lambda \Psi) \gamma^\lambda \Psi = (\bar{\Psi} \Psi) \Psi - (\bar{\Psi} \gamma_5 \Psi) \gamma_5 \Psi.$$

Multiplying by $\bar{\Psi}$ and using the above definitions yields

$$J^\mu J_\mu = A^2 + B^2, \quad (2)$$

where $A \equiv \bar{\Psi} \Psi$ and $B \equiv i\bar{\Psi} \gamma^5 \Psi$. We also have

$$(\bar{\Psi} \gamma_5 \gamma_\lambda \Psi) \gamma^\lambda \Psi = (\bar{\Psi} \gamma_5 \Psi) \Psi - (\bar{\Psi} \Psi) \gamma_5 \Psi.$$

From which it follows that the norm of I_μ is

$$I^\mu I_\mu = -A^2 - B^2 \quad (3)$$

and that the four-vector currents are orthogonal

$$I_\mu J^\mu = 0. \quad (4)$$

It follows that the current J_μ is a time-like vector; and the axial current is space-like.

1. Internal connection

It is useful to treat the equation of motion of the fundamental spinors in a non-euclidean system of coordinates. In order to deal with the covariance of the theory it is necessary to introduce the concept of internal connection. In the case of an arbitrary Riemannian geometry (of which the Minkowski metric is a particular case) Fock and Ivanenko displayed the main properties needed to obtain such covariant description in the case of a spinor. This means, exchanging the simple derivative for a covariant one defined by

$$\nabla_\mu \Psi = \partial_\mu \Psi - i\Gamma_\mu \Psi. \quad (5)$$

In the same way the elements of the Clifford algebra must be conveniently modified. Suppose that we are dealing with the Minkowski geometry in a spherical coordinate system, as is the case later on. The metric takes the form

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2. \quad (6)$$

In consequence, the γ_μ 's are given in terms of the constant $\tilde{\gamma}_\mu$ as follows:

$$\begin{aligned} \gamma_0 &= \tilde{\gamma}_0 \\ \gamma_1 &= \tilde{\gamma}_1 \\ \gamma_2 &= r \tilde{\gamma}_2 \\ \gamma_3 &= r \sin \theta \tilde{\gamma}_3. \end{aligned}$$

For later use we display our convention of the constant γ_μ 's:

$$\tilde{\gamma}^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

$$\tilde{\gamma}^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}.$$

This form was obtained by using the property

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} \mathbb{I}. \quad (7)$$

Note that from now on we will write simply 1 to represent the identity of the Clifford algebra.

In the case of the original Fock-Ivanenko condition (i.e., vanishing of the covariant derivative of the γ_μ) one obtains the form for the FI connection:

$$\Gamma_\mu^0 = \frac{1}{8} [\gamma^\alpha \gamma_{\mu,\alpha} - \gamma_{\mu,\alpha} \gamma^\alpha + \Gamma_{\mu\nu}^\epsilon (\gamma^\epsilon \gamma^\nu - \gamma^\nu \gamma^\epsilon)]. \quad (8)$$

The index 0 in Γ_μ is just a reminder that we are dealing with a Minkowski background in an arbitrary system of coordinates. We can globally annihilate such connection by moving to an Euclidean constant coordinate system.

2. Generalized internal connection

The expression of the internal connection as displayed by Fock and Ivanenko was obtained by assuming that the covariant derivative of all γ_μ vanish. This is a direct consequence of relation (7). Indeed, $\nabla_\mu \gamma_\nu = 0$ implies that the metric is Riemannian: $\nabla_\mu g_{\alpha\beta} = 0$. However, although the condition of vanishing covariant derivatives of γ_μ is enough to guarantee the Riemannian structure of the geometry, it is not necessary. In [6] a case is examined in which the dynamics of the Clifford structure is driven by the condition of the commutator:

$$\nabla_\mu \gamma_\nu = [U_\mu, \gamma_\nu], \quad (9)$$

where U_μ is an arbitrary element of the Clifford algebra.

Indeed, from the relation (7) and using the above expression with $U_\mu = A_\mu + B_\mu \gamma_5$, we have for arbitrary vectors A_μ and B_μ :

$$\nabla_\mu \gamma_\nu = [A_\mu + B_\mu \gamma_5, \gamma_\nu]. \quad (10)$$

We have

$$\nabla_\mu g_{\alpha\beta} = [U_\mu, \gamma_\alpha] \gamma_\beta + \gamma_\alpha [U_\mu, \gamma_\beta] + [U_\mu, \gamma_\beta] \gamma_\alpha + \gamma_\beta [U_\mu, \gamma_\alpha]$$

and using the property that γ_5 anti-commutes with all γ_ν , it follows that $\nabla_\mu g_{\alpha\beta} = 0$. This holds for arbitrary vectors A_μ and B_μ .

We shall see that the internal connection obtained in this way provides an equivalent way to describe the non linear structure of Heisenberg spinors for a convenient choice of U_μ . Thus the internal connection takes the form

$$\Gamma_\mu = \Gamma_\mu^0 - iU_\mu. \quad (11)$$

III. HEISENBERG QUARTIC SELF-INTERACTING SPINORS

3. Historical comment

In a series of papers (see [4] for a complete list) Heisenberg examined a proposal regarding a complete quantum

theory of fields and elementary particles. Such a huge and ambitious program did not fulfill his initial expectation. It is not our intention here to discuss this program. For our purpose, it is important only to retain the original non linear equation of motion which Heisenberg postulated for the constituents of the fundamental material blocks of all existing matter. The modern point of view has developed in a very different direction and it is sufficient to take a look at the book of Particle Data Properties [8] and the description of our actual knowledge of the elementary particle properties to realize how far from Heisenberg dream the theory has gone.

So much for the historical context. What we would like to retain from Heisenberg's approach reduces exclusively to his suggestion of a non linear equation of motion for a spinor field. We will use this equation for both our fundamental spinors, once as we will now see, it is the simplest non-linear dynamics that can be constructed in a covariant way. Let Ψ and Υ be the fundamental four-component spinor field. The dynamics of Ψ (resp., Υ) is given by the self-interaction Lagrangian (we are using the conventional units were $\hbar = c = 1$) :

$$L = \frac{i}{2} \bar{\Psi} \gamma^\mu \partial_\mu \Psi - \frac{i}{2} \partial_\mu \bar{\Psi} \gamma^\mu \Psi - V(\Psi). \quad (12)$$

The potential V is constructed with the two scalars that can be formed with Ψ , that is A and B . We will only consider the Heisenberg potential that is

$$V = s (A^2 + B^2) \quad (13)$$

where s is a real parameter of dimension $(length)^2$.

This potential can be written in an equivalent and more suggestive form in terms of the associated currents J_μ and I_μ . These two vectors will become the basic ingredients of the model which we will deal with in the present paper. As we have anticipated above in equation (2), the Heisenberg potential V is nothing but the norm of the four-vector current J^μ .

The Heisenberg non linear equation of motion:

$$i\gamma^\mu \partial_\mu \Psi - 2s(A + iB\gamma_5)\Psi = 0, \quad (14)$$

follows from the Lagrangian (12) with the potential $V = sJ^2$.

IV. GEOMETRICAL REALIZATION OF THE HEISENBERG SPINOR

In this section we show how to understand the self-coupling of equation (14) in terms of a modification of the internal connection. In so doing, we are preparing our analysis for the universal gravitational interaction of the non-linear spinor theory. Let us use the form (11) and set

$$\Gamma_\mu = -i(aJ_\mu + bI_\mu)(\mathbb{I} + \gamma^5) \quad (15)$$

in which, for simplicity we use an Euclidean coordinate system in which the Fock-Ivanenko standard part of the connection vanishes. Thus, the Lagrangian of the fundamental spinor takes the form

$$\begin{aligned} L &= \frac{i}{2} \bar{\Psi} \gamma^\mu \nabla_\mu \Psi - \frac{i}{2} \nabla_\mu \bar{\Psi} \gamma^\mu \Psi \\ &= \frac{i}{2} \bar{\Psi} \gamma^\mu \partial_\mu \Psi + \frac{1}{2} \bar{\Psi} \gamma^\mu \Gamma_\mu \Psi + h.c. \end{aligned} \quad (16)$$

Substituting the form (15) in this Lagrangian we obtain

$$L = \frac{i}{2} \bar{\Psi} \gamma^\mu \partial_\mu \Psi - \frac{i}{2} \partial_\mu \bar{\Psi} \gamma^\mu \Psi - \frac{i}{2} [(a - \bar{a}) - (b - \bar{b})] J_\mu J^\mu. \quad (17)$$

This is precisely the expression of Heisenberg Lagrangian (12) which led us to the identification

$$s = \frac{i}{2} [(a - \bar{a}) - (b - \bar{b})]. \quad (18)$$

Thus we succeeded to present Heisenberg self-interaction as a modification of the internal connection structure.

4. Two fundamental spinors

Our theory contains two spinors that obey Heisenberg type of equations of motion. Once we can describe such self-coupling in terms of a modification of the internal connection, this procedure automatically implies a direct interaction between Ψ and Υ . We have

$$U_\mu = [a(J_\mu + j_\mu) + b(I_\mu + i_\mu)](1 + \gamma^5). \quad (19)$$

Besides the self-interaction terms there appears a direct interaction between the two spinors given by

$$\begin{aligned} L_{int} &= \frac{i}{2} [(a - \bar{a})(J_\mu + j_\mu) + (b - \bar{b})(I_\mu + i_\mu)] \\ &\quad [J^\mu + j^\mu + I^\mu + i^\mu]. \end{aligned}$$

We set

$$b - \bar{b} = \beta(a - \bar{a}). \quad (20)$$

Thus, for the total interaction Lagrangian we find

$$\begin{aligned} L_{int} &= \frac{i}{2} (a - \bar{a})(1 - \beta)(J_\mu J^\mu + j_\mu j^\mu) \\ &\quad + i(a - \bar{a})\{J_\mu j^\mu + \beta I^\mu i_\mu\} \\ &\quad + \frac{i}{2} (a - \bar{a})(1 + \beta)(J^\mu i_\mu + I^\mu j_\mu). \end{aligned} \quad (21)$$

The first two terms represent the Heisenberg self-interactions and the other terms the interaction between Ψ and Υ . It seems worthwhile to remark that in case $\beta = 1$, the Heisenberg terms vanishes and the interaction assumes the reduced form

$$L_F = g_F \bar{\Psi} \gamma^\mu (1 + \gamma^5) \Psi \bar{\Upsilon} \gamma_\mu (1 + \gamma^5) \Upsilon.$$

where $g_F \equiv (\hbar c)i(a - \bar{a})$.

V. THE UNIVERSAL COUPLING: GRAVITY

Half a century has already elapsed since the idea of dealing with the content of General Relativity in terms of a field theory propagating in a non-observable Minkowski background was presented by Gupta, Feynman and others. In recent times this approach has been revised and commented (see [3] and references therein).

The field theoretical approach goes back to the fact that Einstein dynamics of the curvature of the Riemannian metric of space-time can be obtained as a sort of iterative process, starting from a linear theory of a symmetric second order tensor $\varphi_{\mu\nu}$ and by an infinite sequence of self-interacting process leading to a geometrical description. The definition of the metric is provided in terms of the metric of the background $\eta_{\mu\nu}$ as follows

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + \varphi_{\mu\nu} \quad (22)$$

Note that this is not an approximation formula but an exact one. The inverse metric $(g_{\mu\nu})^{-1} \equiv g^{\mu\nu}$ is defined by $g_{\nu\mu}g^{\mu\alpha} = \delta_\nu^\alpha$. Other definitions were also used, for instance,

$$\sqrt{-g}g^{\mu\nu} \equiv \sqrt{-\gamma}(\gamma^{\mu\nu} + \varphi^{\mu\nu})$$

where $\gamma_{\mu\nu}$ is the background Minkowski metric, written in an arbitrary system of coordinates (see for instance [3] for an analysis of the convenience of these alternative non-equivalent definitions).

Although these theories can be named "field theories" they contain the same metric content of General Relativity, disguised in a non geometrical form. The framework which will be discussed here is totally different. It is important to emphasize that we are not presenting a dynamics for the metric in the sense of such field theories. Instead, the geometry is understood as an effective one, in the sense that it is the way gravity appears for all forms of matter and energy. However, its evolution is given by the fundamental spinor fields Ψ and Υ . We learn from these field theories of gravitation the way to couple the tensor field $\varphi_{\mu\nu}$ with matter terms in order to guarantee that the net effect of this interaction produces the modification of the metric structure. This idea will guide us when coupling the two fundamental spinors with all forms of matter and energy in order to arrive at the same equivalent interpretation of the identification of the gravitational field with the metric of the space-time.

VI. THE UNIVERSAL COUPLING OF Ψ AND Υ WITH MATTER

From the previous section, the reader understands that our strategy is to treat the interaction of the spinors fields in terms of a modification of an internal connection. Now we face the question: how does matter of any form and any kind of energy interact with these two fields? Following this strategy we make a major hypothesis (which

substitutes the corresponding hypothesis made by Einstein on the dynamics of $g_{\mu\nu}$) that the spinors interact universally with all forms of matter/energy through the modification of the internal connection Γ_μ . Let us review briefly the way GR describes this coupling and compare it with our procedure.

Let L_0 be the Lagrangian of a certain matter distribution in the absence of gravitational forces given by

$$S_0 = \int \sqrt{-\gamma} L_0 = \int \sqrt{-\gamma} B^{\mu\nu} \gamma_{\mu\nu}.$$

In order to couple the matter, described by this Lagrangian, with gravity the procedure in the field theory formulation of General Relativity is made through the use of the Equivalence Principle or, as sometimes it is named, the minimal coupling principle. This means substituting of all the terms in the action S_0 in which the Minkowski metric $\gamma_{\mu\nu}$ appears by the general metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. Let us give two examples. First we consider the case of a scalar field Φ . In the Minkowski background its dynamics is provided by

$$S_0 = \int \sqrt{-\gamma} \partial^\mu \Phi \partial_\mu \Phi \gamma_{\mu\nu}.$$

In this case $B^{\mu\nu}$ can be written in terms of the energy-momentum tensor defined as

$$T_{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta}{\delta \gamma^{\mu\nu}} (\sqrt{-\gamma} L).$$

Indeed, a direct calculation yields

$$T^{\mu\nu} = \partial_\alpha \Phi \partial_\beta \Phi \gamma^{\alpha\mu} \gamma^{\beta\nu} - \frac{1}{2} \partial_\lambda \Phi \partial_\sigma \Phi \gamma^{\lambda\sigma} \gamma^{\mu\nu}$$

immediately implying the expression

$$B^{\mu\nu} = T^{\mu\nu} - \frac{1}{2} T \gamma^{\mu\nu},$$

where $T \equiv T^{\mu\nu} \gamma_{\mu\nu}$. The corresponding action, including the gravitational interaction, is obtained by changing all $\gamma_{\mu\nu}$ and its inverse $\gamma^{\mu\nu}$ with the corresponding $g_{\mu\nu} = \gamma_{\mu\nu} + \varphi_{\mu\nu}$ which yields

$$S = \int \sqrt{-g} \partial^\mu \Phi \partial_\mu \Phi g_{\mu\nu},$$

where $g = \det g_{\mu\nu}$. In this case

$$B^{\mu\nu} = \frac{\sqrt{-g}}{\sqrt{-\gamma}} [T^{\mu\nu} - \frac{1}{2} T g^{\mu\nu}].$$

Let us consider now the case of the electromagnetic field. The action is given by

$$S_0 = \int \sqrt{-\gamma} F^{\alpha\mu} F^{\beta\nu} \gamma_{\alpha\beta} \gamma_{\mu\nu}$$

The same reasoning yields

$$B^{\mu\nu} \equiv F^{\alpha\mu} F^{\beta\nu} \gamma_{\alpha\beta}$$

and we follow the same rules as in the previous scalar equation of motion. The fact that it is not possible to write the tensor $B^{\mu\nu}$ in terms of the energy-momentum tensor in this case is due to the fact that this tensor is traceless.

Let us turn now to the analogous analysis in the Spinor Theory of Gravity. We follow a similar procedure. Our strategy is to modify the internal connection and change U_μ by the form:

$$U_\mu = U_\mu^1 + U_\mu^2 \quad (23)$$

where

$$U_\mu^1 = [a(J_\mu + j_\mu) + b(I_\mu + i_\mu)] (1 + \gamma_5)$$

and

$$U_\mu^2 = -\frac{\lambda}{\sqrt{X}} [a(J_\alpha + j_\alpha) + b(I_\alpha + i_\alpha)] B_\mu^\alpha (1 + \gamma_5)$$

where X is the sum of the norms of the vectors J_μ and j_μ , i.e. $X = J_\mu J^\mu + j_\mu j^\mu$ and λ is a constant of dimension $(\text{energy})^{-1}$. The first term U_μ^1 represents the free-field and the second one the interaction of the fundamental spinors with matter.

In the present theory this is how matter affects the spinor fields. To be explicit, let us write the complete Lagrangian for one of the spinors (analogous form corresponds to the other one). Taking into account the fact that $B_{\mu\nu}$ is symmetric and inserting U_μ^2 into equation (16), we find for the matter interaction part

$$L_g = -g_F \frac{\lambda}{4} \frac{(c_{\mu\nu} + c_{\nu\mu})}{\sqrt{X}} B^{\mu\nu} \quad (24)$$

where

$$c_{\mu\nu} = [J_\mu + j_\mu + I_\mu + i_\mu][J_\nu + j_\nu + \beta(I_\nu + i_\nu)]$$

This form of interaction of matter/energy with the fundamental spinor fields leads to the definition of a metric, in the same way as in the field theory representation of General Relativity namely

$$g_{\mu\nu} = \eta_{\mu\nu} + \varphi_{\mu\nu} \quad (25)$$

where the field $\varphi_{\mu\nu}$ is chosen to be non-dimensional, that is, we set:

$$\varphi_{\mu\nu} = -\frac{g_F \lambda}{4} \frac{1}{\sqrt{X}} (c_{\mu\nu} + c_{\nu\mu}) \quad (26)$$

Due to our choice of the definition of the effective metric $g_{\mu\nu}$ it follows that $B^{\mu\nu}$ is the same as in the field theory representation of General Relativity presented above. This way of coupling matter with the fundamental spinors guarantees that in what concerns the behavior of matter the analysis of General Relativity is still valid in the present theory: free particles follow geodesics in the effective geometry $g_{\mu\nu}$. The most important task now is to analyze the consequences of this theory. We start this by studying the effective metric generated by the gravitational process in the neighborhood of a massive object, like a star.

VII. GRAVITATIONAL FIELD OF A COMPACT OBJECT

In the absence of matter and energy, the effective metric can be obtained by a direct solution of the Heisenberg equation and the identification of $g_{\mu\nu}$ through equation (25). The equations of motion in this case are

$$\begin{aligned} i\gamma^\mu \partial_\mu \Psi + \gamma^\mu \Gamma_\mu^{(0)} \Psi & - 2s(A + iB\gamma^5) \Psi \\ & - g_F \gamma^\mu \left(j_\mu + \frac{1+\beta}{2} i_\mu \right) \Psi \\ & - g_F \left(\beta i_\mu + \frac{1+\beta}{2} j_\mu \right) \gamma^\mu \gamma^5 \Psi = 0. \end{aligned} \quad (27)$$

$$\begin{aligned} i\gamma^\mu \partial_\mu \Upsilon + \gamma^\mu \Gamma_\mu^{(0)} \Upsilon & - 2s(\hat{A} + i\hat{B}\gamma^5) \Upsilon \\ & - g_F \gamma^\mu \left(J_\mu + \frac{1+\beta}{2} I_\mu \right) \Upsilon \\ & - g_F \left(\beta I_\mu + \frac{1+\beta}{2} J_\mu \right) \gamma^\mu \gamma^5 \Upsilon = 0. \end{aligned} \quad (28)$$

where $\hat{A} \equiv \bar{\Upsilon} \Upsilon$ and $\hat{B} \equiv i\bar{\Upsilon} \gamma^5 \Upsilon$.

This is a highly non linear system that must be solved in order to obtain the effective metric. We succeeded in finding a solution in the case of a spherically symmetric and static configuration. Using the background Minkowski metric in the form (6) we obtain the unique non identically background FI connection:

$$\begin{aligned} \Gamma_2^{(0)} &= \frac{1}{2} \tilde{\gamma}_1 \tilde{\gamma}_2 \\ \Gamma_3^{(0)} &= \frac{1}{2} \sin\theta \tilde{\gamma}_1 \tilde{\gamma}_3 + \frac{1}{2} \cos\theta \tilde{\gamma}_2 \tilde{\gamma}_3 \end{aligned}$$

We will look for a solution of the form

$$\Psi = f(r) e^{i\varepsilon \ln r} e^{ih(\theta)} \Psi^0$$

$$\Upsilon = g(r) e^{i\tau \ln r} e^{il(\theta)} \Upsilon^0$$

where ε and τ are constants; Ψ^0 and Υ^0 are constant spinors. The Heisenberg equation of motion is solved if $h(\theta)$ and $l(\theta)$ are proportional to $\ln \sqrt{\sin\theta}$. Moreover, $f(r)$ and $g(r)$ obey the equations

$$\frac{1}{f^3} \frac{df}{dr} = \text{constant},$$

$$\frac{1}{g^3} \frac{dg}{dr} = \text{constant}.$$

We then have

$$\Psi = \frac{u}{\sqrt{r}} e^{i\varepsilon \ln r} e^{iln\sqrt{\sin\theta}} \Psi^0, \quad (29)$$

$$\Upsilon = \frac{u'}{\sqrt{r}} e^{i\tau \ln r} e^{i \ln \sqrt{\sin \theta}} \Upsilon^0. \quad (30)$$

for constants u and u' . The dependence on the angle θ disappears in both (vector and axial) currents. The $r^{-\frac{1}{2}}$ term depends on the fact that the Heisenberg potential is of quartic order. Any other dependence should yield a different functional dependence for the effective metric. As we shall see next, this form is crucial in order to obtain the good behavior of the metric in the newtonian limit.

We set

$$\Psi^0 = \begin{pmatrix} \varphi^0 \\ \eta^0 \end{pmatrix} \quad (31)$$

To solve the equation of motion, the constant spinor Ψ^0 (correspondingly Υ^0) must satisfy a set of equations. We set

$$\varphi^0 = (c_1 + c_2 \sigma_1) \eta^0 \quad (32)$$

We look for a solution such that

$$\sigma_1 \eta^0 = \eta^0.$$

which yields

$$\varphi^0 = i R \eta^0,$$

where R is a real number. Note that all currents from the expression of Ψ and Υ are of the form a^μ/r for different constant vectors a^μ . After a rather long and tedious calculation we obtain the final expressions of these currents constructed with our solution. It is precisely these currents that provide the effective metric, namely:

$$\begin{aligned} J_o &= \frac{p}{r} \\ I_o &= \frac{q}{r} \\ J_1 &= \frac{m}{r} \\ I_1 &= \frac{n}{r} \end{aligned}$$

and analogous formulas for the spinor Υ :

$$\begin{aligned} j_o &= \frac{p'}{r} \\ i_o &= \frac{q'}{r} \\ j_1 &= \frac{m'}{r} \\ i_1 &= \frac{n'}{r} \end{aligned}$$

where

$$\begin{aligned} p &= [c_1 \bar{c}_1 + c_2 \bar{c}_2 + 1] \eta^+ \eta + [c_1 \bar{c}_2 + c_2 \bar{c}_1] \eta^+ \sigma_1 \eta \\ q &= -[c_1 + \bar{c}_1] \eta^+ \eta - [c_2 + \bar{c}_2] \eta^+ \sigma_1 \eta \\ m &= [c_2 + \bar{c}_2] \eta^+ \eta + [c_1 + \bar{c}_1] \eta^+ \sigma_1 \eta \\ n &= [c_1 \bar{c}_2 + c_2 \bar{c}_1] \eta^+ \eta + [c_1 \bar{c}_1 + c_2 \bar{c}_2 + 1] \eta^+ \sigma_1 \eta \end{aligned}$$

Similar formulas holds for the corresponding quantities constructed with Υ involving p', q', m', n' . Analogously we set

$$\Upsilon^0 = \begin{pmatrix} \chi^0 \\ \zeta^0 \end{pmatrix} \quad (33)$$

and:

$$\chi^0 = (d_1 + d_2 \sigma_1) \zeta^0 \quad (34)$$

where

$$\sigma_1 \zeta^0 = -\zeta^0,$$

and

$$\chi^0 = i S \zeta^0$$

where S is a real number.

Since the constants c_1, c_2, d_1 and d_2 are purely imaginary numbers it follows that $m = q = m' = q' = 0$. Consistency imposes the four conditions

$$\varepsilon = 1 - \frac{g_F}{2} (1 + \beta)(n' + p'), \quad (35)$$

$$\tau = 1 - \frac{g_F}{2} (1 + \beta)(n + p) \quad (36)$$

$$-2s u^2 A_0 + 2s R u^2 B_0 + g_F(p' + \beta n') - \frac{1}{2} R = 0 \quad (37)$$

$$2s(u)^2(RA_0 + B_0) + Rg_F(p' + \beta n') + \frac{1}{2} = 0. \quad (38)$$

By symmetry, the components (2) and (3) of the currents $J_\mu, I_\mu, j_\mu, i_\mu$, must vanish. This is possible if the constant spinors satisfy:

$$\begin{aligned} \eta_0^+ \sigma_2 \eta_0 &= 0 \\ \eta_0^+ \sigma_3 \eta_0 &= 0 \end{aligned} \quad (39)$$

and

$$\begin{aligned} \zeta_0^+ \sigma_2 \zeta_0 &= 0 \\ \zeta_0^+ \sigma_3 \zeta_0 &= 0 \end{aligned} \quad (40)$$

Once all these conditions are satisfied there remains two arbitrary conditions to be fixed, for instance $\eta_0^+ \eta_0$ and $\zeta_0^+ \zeta_0$. Different choices yield different solutions for the spinor fields and consequently distinct configurations for the observable metric.

VIII. THE EFFECTIVE METRIC

From the above solution of the spinor fields we can evaluate the currents and the effective geometry that acts on all forms of matter and energy. From its dependence on r and θ we have that all currents depend only on $1/r$. Using the expression of the effective metric in terms of the spinorial fields provided by equations (25) and (26), a direct calculation gives:

$$ds^2 = \left(1 - \frac{r_H}{r}\right) dt^2 + 2 \frac{N}{r} dr dt - \left(1 + \frac{Q}{r}\right) dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2, \quad (41)$$

where:

$$\begin{aligned} r_H &= \frac{g_F \lambda}{2} \frac{1}{\sqrt{Z}} (p + p')^2, \\ Q &= \frac{g_F \lambda}{2} \frac{1}{\sqrt{Z}} \beta (n + n')^2, \\ N &= -\frac{g_F \lambda}{4} \frac{1}{\sqrt{Z}} (p + p')(n + n'). \end{aligned}$$

The constant Z is defined in terms of the norm of the currents as $Z = X r^2 = p^2 + (p')^2$.

In order to compare this geometry with the corresponding solution in General Relativity, we make a co-ordinate transformation to eliminate the crossing term $dr dt$. Setting

$$dt = dT + \frac{N}{r - r_H} dr,$$

we obtain

$$ds^2 = \left(1 - \frac{r_H}{r}\right) dT^2 - \left(1 - \frac{r_H}{r}\right)^{-1} \left(1 - \frac{r_H - Q}{r} - \frac{Q r_H - N^2}{r^2}\right) dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2. \quad (42)$$

At this point we remark that in the case of General Relativity, Birkhoff's theorem forbids the existence of more than one arbitrary constant in the Schwarzschild solution. In the present case of the Spinor Gravity theory, this theorem does not apply. Thus we can understand the fact that this solution contains one additional arbitrary constant. Observations [9] impose that for small values of r_H/r the factors g_{00} and g_{11} must be in the first order respectively $g_{00} = 1 - r_H/r$ and $g_{11} = -1 - r_H/r$. This fact imply that the the constants $\eta_0^+ \eta_0$ and $\zeta_0^+ \zeta_0$ must be chosen such that $r_H = Q$. This fixes one constant. The other constant is provided, as in the similar procedure in GR, by the newtonian limit for $r \rightarrow \infty$, in terms of the Newton constant and the mass of the compact object that is, $r_H = 2g_N M/c^2$. Thus, the final form of the

effective metric is given by

$$ds^2 = \left(1 - \frac{r_H}{r}\right) dT^2 - \left(1 - \frac{r_H}{r}\right)^{-1} \left[1 + \sigma^2 \left(\frac{r_H}{r}\right)^2\right] dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2, \quad (43)$$

where

$$\sigma^2 \equiv \frac{(\beta - 1)^2}{4\beta}.$$

It is a remarkable consequence of the above solution that in the case in which the self-interaction of the fundamental spinors vanishes and only the interaction between Ψ and Υ occurs, that is, for $\beta = 1$ the four-geometry is precisely the same as the Schwarzschild solution in GR. On the other hand, if $\beta \neq 1$ the difference between both theories appears already in the order $(r_H/r)^2$. Indeed for General Relativity we have

$$-g_{11} = 1 + \frac{r_H}{r} + \left(\frac{r_H}{r}\right)^2$$

and for the Spinor Theory we obtain

$$-g_{11} = 1 + \frac{r_H}{r} + \left(\frac{r_H}{r}\right)^2 (1 + \sigma^2).$$

The parameter β should be fixed by observation.

IX. CONCLUSION

In the present paper we have presented a new formalism to describe gravity. We have shown that there is an alternative way to implement the Equivalence Principle in which the geometry acting on matter is not an independent field, and as such does not posses its own dynamics. Instead, it inherits one from the dynamics of two fundamental spinor fields Ψ and Υ which are responsible for the gravitational interaction and through which the effective geometry appears. We have presented a specific model by using Heisenberg equation of motion for the self-interacting spinors. This equation of motion can be understood in terms of a modification of the internal connection as seen by Ψ and Υ and only by these two spinors. This dynamics, which involves not only the self terms but also a specific coupling among these two fields, provides an evolution for the effective metric (constructed in terms of these spinors) which is the way the fields Ψ and Υ interact with all other forms of matter and energy. We have succeeded in finding a solution for the fields, and thus we can extract the behavior of the effective metric in the case of a static spherically symmetric configuration. The result is similar as in General Relativity, showing the existence of horizon and the possibility of existence of Black Hole. They are not identical and differ already on the order $(r_H/r)^2$.

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